

# Degradation Analysis of Probabilistic Parallel Choice Systems

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## Abstract

Degradation analysis is used to analyze the useful lifetimes of systems, their failure rates, and various other system parameters like mean time to failure (MTTF), mean time between failures (MTBF), and the system failure rate (SFR). In many systems, certain possible parallel paths of execution that have greater chances of success are preferred over others. Thus we introduce here the concept of *probabilistic parallel choice*. We use binary and  $n$ -ary probabilistic choice operators in describing the selections of parallel paths. These binary and  $n$ -ary probabilistic choice operators are considered so as to represent the complete system (described as a series-parallel system) in terms of the probabilities of selection of parallel paths and their relevant parameters. Our approach allows us to derive new and generalized formulae for system parameters like MTTF, MTBF, and SFR. We use a generalized exponential distribution, allowing distinct installation times for individual components, and use this model to derive expressions for such system parameters.

*Keywords:* reliability block diagram (RBD); reliability time estimation (RTE); mean time to failure (MTTF); mean time between failures (MTBF); mean time to repair (MTTR); system failure rate (SFR); probability density function (pdf); probabilistic choice; non-probabilistic choice.

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## Notation

$X_i$	: Component with index $i$
$P(X_i)$	: Probability that component $X_i$ works successfully
$P_\delta(X_i)$	: Probability that component $X_i$ fails
$R_k$	: Parallel path $k$
$P(R_k)$	: Probability that parallel path $R_k$ works successfully
$P_\delta(R_k)$	: Probability that parallel path $R_k$ fails
$S$	: A complete system
$P(S, t \geq T)$	: Probability of success of system $S$ for $t \geq T$
$P_\delta(S, t \geq T)$	: Probability of failure of system $S$ for $t \geq T$
$\oplus$	: Non-probabilistic choice operator
$\otimes$	: Sequential choice operator
$\bar{\wedge}_\psi$	: Probabilistic choice operator (binary case)
$\bar{\wedge}_{\psi_k}$	: $k$ th Probabilistic choice operator ( $n$ -ary case)
$\psi_k$	: Probability with which parallel path $R_k$ is chosen
$t_{0i}$	: Installation time of component $X_i$
$t_\infty$	: Time at which probability of successful working of a system is almost zero
$\rho$	: Threshold value of probability of success of system for system to be reliable in determination of mean time to failure
$\lambda_i$	: Failure rate of component $X_i$ which determines how fast the probability of success decays with time
$\lambda_{eq}$	: The equivalent system failure rate given components with $\lambda_i$ individual failure rates

## 1 Introduction

Degradation analysis is used to determine system parameters related to reliability, and is commonly done using reliability block diagrams (RBDs), assuming time-dependent distributions like exponential, Weibull, normal, etc. We propose a model to compute more generalized formulae for these parameters, and indicate such generalized formulae for the exponential distribution in particular. Traditional modelling techniques such as RBDs treat each component as a node in a diagram; several such nodes combine in series or parallel combinations to form a system. Each node is associated with a time-dependent probability distribution. The probabilities of success and failure of these components are assumed to vary according to the distribution considered. Further, the system parameters are calculated based on the

distribution considered.

A major accomplishment of this work lies in development of new and better formulae for mean time to failure (MTTF), system failure rate (SFR), mean time between failures (MTBF), reliability time estimation (RTE), probability density function (pdf), and mean time to repair (MTTR), using the new approach of probabilistic parallel choice. Like the series-parallel systems considered in classical works such as Bazovsky [2] and Birolini [3], we assume each complex system to be composed of series and parallel combinations of components. We use probabilistic choice operators to represent a system having components in parallel. The probabilistic choice in selection of parallel paths extends the traditional model for systems, as the probabilities of selection of parallel paths are not equally likely (as is implicitly the case in all prior works).

There are four operators in our model. These include probabilistic choice operators for both binary and  $n$ -ary cases (compare with Andova [1]). Two other operators, the non-probabilistic choice operator and the sequential operator, are also used. Classical RBDs use a non-probabilistic choice operator in considering selection among parallel paths, so that all the parallel paths are assumed to be identical and each paths selection is considered equally likely. This assumption in RBDs and related models is certainly a limitation, as some systems may choose certain paths with higher probabilities than others, e.g., when a system is designed so that a more reliable—or less expensive—path shall be chosen prior to, or more often than, another.

The relevance of probabilistic parallelism thus lies in the fact that parallel paths are not identical in many real systems. There can be some associated advantages and disadvantages in traversing particular paths: a particular path may be faster but at the same time costlier, or it may be safer or less disruptive than another path.

A binary probabilistic choice operator as in Andova [1] applies in case of two paths in parallel where there is a probabilistic choice attached to the selection of each parallel path. An  $n$ -ary probabilistic choice operator, also as in Andova [1] applies with  $n$  paths in parallel, if there exists a probabilistic choice in selection of each of the parallel paths. Later, in Section 2.4 we present a method to quantify the probability  $\psi_k$  of selection of a given path  $R_k$ . The probabilistic choice operators are used to represent two or more paths in parallel, with distinct probabilities of selection attached to all paths. Unlike in RBDs, each path in parallel is treated differently, and is given a priority based on the chances of success associated with it. This allows us to come up with a better degradation analysis of systems. The series system is similar to that described in Gottumukkala [5], Bazovsky [2]

and Birolini [3], and uses a sequential operator to represent consecutive components or subsystems in series. In Section 3, we introduce time-dependent probabilities of success and failure using the exponential distribution. In our analyses, each component is assumed to have a probability of success and failure that varies exponentially with time.

It is well known that when a system breaks down, not all components of the system are considered as faulty. Replacement or repair is generally considered essential only for those components that have actually ceased to work properly, and have contributed to the current breakdown of the system. Thus, assuming an identical age for each component in performing degradation analysis of a system is usually not a very sound idea—some of the components in the system may have different ages for which they have been operational, and thus their actual probabilities of success and failure can be quite different from that obtained by considering identical ages for all components. We thus assume that each component has a lifetime after which it is to be repaired or replaced, and that after it is repaired or replaced, the component is as good as new. The installation time is the time at which the component is inserted *ab initio*, is repaired, or is replaced. For a time close to an installation time, the chances of successful working of the component are high, and as the time increases the chances of its successful working decay. Earlier models assumed installation times for all components to be fixed and identical [2, 3], but in our model we consider different times of installation for different components in general. Thus the probabilities of success of individual components can be assumed to follow an exponential distribution for times greater than their installation times. Based on this, we derive better formulae in degradation analysis for system parameters like MTTF, SFR, MTBF, MTTR, and pdf. The formulae obtained are more generally applicable compared to previous ones, as earlier work did not incorporate a probabilistic choice in selection of parallel paths, or variable installation times.

## 2 System Model

A *system* in our model comprises of a series and parallel combination of independently-working units called *components*. These components have attached probabilities of success and failure. In general, these probabilities may be constant or time-dependent, depending on the type of distribution followed. We also assume that all components in a system are independently-working units of that system, so the success or failure of any component is

independent of the success or failure of other components.

The probability of success of a system is a measure of its overall performance. For each component  $X_i$ , the probability of successful working of the component is given by  $P(X_i)$  and a probability of failure of the component given by  $P_\delta(X_i) = 1 - P(X_i)$ . The success or failure of the complete system is dependent upon the success and failure of the individual components of which it is composed.

The system is assumed to be repairable system, that is, replacement or repair of the components is undertaken if they cease to work causing failure of the entire system. Any component so repaired or replaced is assumed to be as good as a new one.

In our model, we assume that each parallel path  $R_k$  is chosen with a different probability  $\psi_k$ , and that these selection probabilities of parallel paths remain even though the probabilities of success and failure of individual components in the paths may be vary over time. We further assume (without loss of generality) that each parallel path in a system contains exactly one component.

## 2.1 Related Operators

Our model has four basic operators. These operators are used for representing series and parallel combinations of components.

- Non-probabilistic choice operator ( $\oplus$ ):

$R_1 \oplus R_2$  means that the probability of choosing path  $R_1$  is equal to the probability of choosing path  $R_2$ .

- Sequential operator ( $\otimes$ ):

The sequential operator  $\otimes$  is used such that  $X_1 \otimes X_2$  implies that the probability of choosing  $X_1$  is equal to the probability of choosing  $X_2$ .

- Binary probabilistic choice operator  $\bar{\wedge}_\psi$ :

$R_1 \bar{\wedge}_\psi R_2$  represents a parallel system with parallel paths  $R_1$  and  $R_2$ , where the probability of selection of path  $R_1$  is equal to  $\psi$ , and the probability of selection of path  $R_2$  is equal to  $1 - \psi$ , with  $0 \leq \psi \leq 1$ .

- $n$ -ary probabilistic choice operator ( $\bar{\bar{\wedge}}_{\psi_k}$ ), with  $n \geq 2$ :

$R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n$  represents a parallel system with  $n$  parallel paths  $R_1$  through  $R_n$ , where the probability of choosing path  $R_k$  is  $\psi_k$ ,  $1 \leq k \leq n-1$ , and the probability of choosing path  $R_n$  is  $1 - \sum_{k=1}^{n-1} \psi_k$ .

We now present a basic theorem relating the binary and  $n$ -ary probabilistic choice operators. It is useful in calculating the overall probability of success or failure of complex systems in terms of the relevant parameters of their components.

**Theorem 2.1.** The following hold with respect to  $\bar{\wedge}_\psi$  and  $\bar{\bar{\wedge}}_{\psi_k}$ .

- (a)  $P(R_1 \bar{\wedge}_\psi R_2) = \psi P(R_1) + (1 - \psi)P(R_2)$
- (b)  $P(R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} R_3 \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n)$   
 $= \psi_1 P(R_1) + \psi_2 P(R_2) + \dots + \psi_{n-1} P(R_{n-1})$   
 $+ \left(1 - \sum_{k=1}^{n-1} \psi_k\right) P(R_n)$

*Proof.* For part (a), we note that  $(R_1 \bar{\wedge}_\psi R_2)$  represents a system which has paths  $R_1$  and  $R_2$  as alternatives, with the probability of choosing path  $R_1$  as  $\psi$  and probability of choosing path  $R_2$  as  $1 - \psi$ . Thus the probability of success of the entire system  $(R_1 \bar{\wedge}_\psi R_2)$  can be written with the basic total probability formula:

$$P(A) = \sum_{k=1}^n P(A|X_k)P(X_k) \quad (1)$$

Using (1) we can compute  $P(R_1 \bar{\wedge}_\psi R_2)$  as:

$$P(R_1 \bar{\wedge}_\psi R_2) = P((R_1 \bar{\wedge}_\psi R_2)|R_1)P(R_1) + P((R_1 \bar{\wedge}_\psi R_2)|R_2)P(R_2)$$

Here  $P(R_1)$  and  $P(R_2)$  denote the probabilities of successful working of paths  $R_1$  and  $R_2$ .

From this, we get:

$$P(R_1 \bar{\wedge}_\psi R_2) = \psi P(R_1) + (1 - \psi)P(R_2)$$

Hence part (a) holds.

For part (b), we similarly extend the formula to  $n$  components in parallel, with the probability of selection of the parallel path  $R_k$  being as previously given.

Using the total probability formula (1) we can compute the value of  $P(R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} R_3 \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n)$  as:

$$\sum_{k=1}^{n-1} P((R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} R_3 \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n) | R_k) P(R_k),$$

where  $P(R_k)$  denotes the probability of successful working of path  $R_k$ ,  $\forall k \in 1, 2, \dots, n$ . Thus we get:

$$P(R_1 \bar{\wedge}_{\psi_1} R_2 \bar{\wedge}_{\psi_2} R_3 \dots \bar{\wedge}_{\psi_{n-1}} R_n) = \psi_1 P(R_1) + \psi_2 P(R_2) + \dots + \psi_{n-1} P(R_{n-1}) + \left(1 - \sum_{k=1}^{n-1} \psi_k\right) P(R_n).$$

Hence part (b) holds.  $\square$

In line with our assumption about parallel paths having exactly one component apiece, an obvious corollary may be stated.

**Corollary 2.2.** If each parallel path has exactly one component  $X_i$  in it, the following is true of  $\bar{\wedge}_{\psi}$  and  $\bar{\wedge}_{\psi_i}$ .

$$\begin{aligned} \text{(a)} \quad & P(X_1 \bar{\wedge}_{\psi_i} X_2) = \psi_i P(X_1) + (1 - \psi_i) P(X_2) \\ \text{(b)} \quad & P(X_1 \bar{\wedge}_{\psi_1} X_2 \bar{\wedge}_{\psi_2} X_3 \dots \bar{\wedge}_{\psi_{n-1}} X_n) \\ &= \psi_1 P(X_1) + \psi_2 P(X_2) + \dots + \psi_{n-1} P(X_{n-1}) \\ &+ \left(1 - \sum_{i=1}^{n-1} \psi_i\right) P(X_n) \end{aligned}$$

Every system in this model is assumed to be composed of some series and parallel combination of components. The two basic types of systems, that is, series and parallel systems, are described in Sections 2.2 and 2.3 using operators defined in Section 2.1.

## 2.2 Series System

An  $n$ -component series system with  $n \geq 2$  components  $X_i$  fails if any one of the  $X_i$  fails to work.

$P(X_1), P(X_2), \dots, P(X_n)$  denote the probabilities of success of components  $X_1, X_2, \dots, X_n$ . Thus the probabilities of failure of components  $X_1, X_2, \dots, X_n$  are  $1 - P(X_1), 1 - P(X_2), \dots, 1 - P(X_n)$ .

A series system with  $n$  components  $(X_1, X_2, \dots, X_n)$  can be represented as:

$$X_1 \otimes X_2 \otimes X_3 \dots \otimes X_n.$$

As already mentioned, the components of the system are assumed to be independently working units, and thus the probability of success of the series system is given by:

$$P(X_1 \otimes \dots \otimes X_n) = P(X_1)P(X_2) \dots P(X_n)$$

This formula is similar to that given by Yu and Fuh [9].

The two possible outcomes of the system are either a success or a failure, which are mutually exclusive and cumulatively exhaustive events.

Thus, the probability of failure of the series system, as by Yu and Fuh [9], is given by:

$$P_\delta(X_1 \otimes \dots \otimes X_n) = 1 - P(X_1 \otimes X_2 \otimes \dots \otimes X_n) \quad (2)$$

$$= 1 - P(X_1)P(X_2) \dots P(X_n) \quad (3)$$

The overall probability of success of the series system is smaller than the component with the least probability of success, as overall probability of success of the series system is given by product of probability of success of individual components in series.

A failed component in series with any other component leads to failure of the latter also.

### 2.3 Parallel System

An  $n$ -path parallel system is one in which  $n \geq 2$  parallel paths  $R_j$  are involved, where the system fails only if all the parallel paths  $R_j$  fail to work.

Let  $P(R_1), P(R_2), \dots, P(R_n)$  denote the probabilities of success of parallel paths  $R_1, R_2, \dots, R_n$ . As the two possible outcomes of any parallel path are a successful working of the path or its failure, the probability of failure of paths  $R_1, R_2, \dots, R_n$  is given by  $1 - P(R_1), 1 - P(R_2), \dots, 1 - P(R_n)$  respectively.

#### 2.3.1 Two-Path Parallel System

We first discuss systems which contain exactly two paths in parallel. For such systems the choice is represented using operators  $\oplus$  and  $\bar{\wedge}_\psi$ .

If a process can be executed by either of two parallel paths  $R_1$  or  $R_2$ , that is,  $R_1$  and  $R_2$  are in parallel and both are equally likely in terms of their selection, then this choice is represented as  $R_1 \oplus R_2$  and called non-probabilistic choice. This is similar to the operator used in reliability block diagrams to represent a parallel system.

If a process can be executed by either of two parallel paths  $R_1$  or  $R_2$ , that is,  $R_1$  and  $R_2$  are in parallel and both are chosen with different probabilities of selection, then this choice is represented as  $R_1 \bar{\wedge}_\psi R_2$  and called binary probabilistic choice, as in Andova [1].



### 2.3.2 $n$ -Path Parallel System

We now discuss systems which contain  $n$  parallel paths, where  $n \geq 2$ . For such systems the choice is represented using  $\bar{\bar{\wedge}}_\psi$ .

If a process in a system can be executed by any of the  $n$  parallel paths  $R_k$ ,  $\forall k \in (1, 2, \dots, n)$ , where the probability of selection of path  $R_k$  is given by  $\psi_k$  and  $\sum_{k=1}^n \psi_k = 1$ , then for such a system the choice is represented using  $\bar{\bar{\wedge}}_\psi$  as:

$$R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n. \quad (4)$$

Using Theorem 2.1, the probability of success of the parallel system, which we may denote by  $P(R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n)$ , is given by:

$$\psi_1 P(R_1) + \psi_2 P(R_2) + \dots + (1 - \sum_{k=1}^{n-1} \psi_k) P(R_n). \quad (5)$$

As the two possible outcomes of a system are success and failure, the probability of failure of the parallel system is given by:

$$P_\delta(R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n) = 1 - P(R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n)$$

This leads to:

$$P_\delta(R_1 \bar{\bar{\wedge}}_{\psi_1} R_2 \bar{\bar{\wedge}}_{\psi_2} \dots \bar{\bar{\wedge}}_{\psi_{n-1}} R_n) = 1 - \sum_{k=1}^n \psi_k P(R_k), \quad (6)$$

$$\text{where } \psi_n = 1 - \sum_{k=1}^{n-1} \psi_k.$$

Using Corollary 2.2, we can arrive at an analogous result with  $X_i$  instead of  $R_k$ , when each parallel path has exactly one component.

## 2.4 Assigning The Probability Of Selection Of Parallel Path

$\psi_k$

We now consider a specific method (but obviously not the only one) to assign the probability of selection  $\psi_k$  for a parallel path  $R_k$ , under the assumption that the probability of selection of any path takes into account the probabilities of success and failure of that path.

For any path  $R_k$ , we may have

$$\psi_k \propto \frac{P(R_k)}{P_\delta(R_k)}, \quad (7)$$

and of course also,

$$\sum_{k=1}^n \psi_k = 1. \quad (8)$$

Between (7) and (8), we can calculate and assign values to the  $\psi_k$ .

### 3 Probabilistic Approach To Reliability

It is well known that considerations of probability distributions in reliability analyses allow us to consider time-dependent functioning of systems. The probability of success is assumed to vary with time according to some distribution. This is a reasonable assumption, as the probability of success or failure of any system depends on the time for which it has been used. We assume that all the differently-aged components start working (or attempts are made to start them working) at the same time.

As the exponential distribution is the one most commonly applied in reliability analyses [2, 3], and is thus the basis of a lot of existing theory, we show how to use our approach, with its probabilistic parallel choice operators, to derive expressions for system reliability parameters, assuming this distribution. We note that similar analyses are possible with other distributions as well.

#### 3.1 Exponential Distribution

A general exponential distribution is given by  $\lambda_i \exp(-\lambda_i(t - \hat{t}_{0i}))$ . Unlike other models, e.g., in Yu and Fuh [9], in our analysis we consider  $\hat{t}_{0i}$  as the time of installation of the component  $X_i$ , i.e., that different components of the system may have different times of installation.

The probability of success of component  $X_i$  is considered to vary with time according to the exponential distribution, which of course means that the probability of success of the overall system falls with time, or that the probability of failure of the system increases with time. If  $\lambda_i$  is the rate of failure of the component  $X_i$ , the probability of success of component  $X_i$  at and after time  $T$  is given by:

$$P(X_i, t \geq T) = \int_T^\infty \lambda_i \exp(-\lambda_i(t - \hat{t}_{0i})) dt.$$

This in turn gives:

$$P(X_i, t \geq T) = \exp(-\lambda_i(T - \hat{t}_{0i})). \quad (9)$$

1. *Initially at time  $t = \hat{t}_{0i}$*

$$P(X_i, t) = \exp(-\lambda_i(0)) = 1.$$

This result is in line with our assumption that initially for time close to the installation time, the probability of success of the component is high and is almost one.

2. *As time  $t \rightarrow t_\infty$*

$$\lim_{t \rightarrow \infty} P(X_i, t \approx t_\infty) = \exp(-\lambda_i(\infty)) = 0.$$

This is in line with our assumption that at infinite time, the probability of success of the component  $X_i$  is very low, almost zero.

### 3.2 Application Of This Distribution To Estimation Of System Parameters

#### 3.2.1 Reliability Time Estimation (RTE)

Reliability time is the time until which the probability of success of a system is greater than some minimum required probability of success  $\rho$ . This value  $\rho$  is externally determined in advance according to the desired system performance.

The formulae obtained are general and different from those obtained in Bazovsky [2] and Birolini [3], as our model itself is more general than classical RBDs.

- (i) Generalized  $n$ -component series system:

As the probability of success of series system should be greater than  $\rho$ , we get:

$$P(X_1 \otimes X_2 \otimes X_3 \dots X_n) \geq \rho$$

Using (2) and (9), we get:

$$\rho \leq \exp\left(-\lambda_1(t - \hat{t}_{01})\right) \exp\left(-\lambda_2(t - \hat{t}_{02})\right) \dots \exp\left(-\lambda_n(t - \hat{t}_{0n})\right) \quad (10)$$

Taking the natural logarithm on both sides of inequation (10), we have:

$$t \leq -\frac{\ln(\rho)}{\sum_{i=1}^n \lambda_i} + \frac{\sum_{i=1}^n \lambda_i \hat{t}_{0i}}{\sum_{i=1}^n \lambda_i}.$$

(ii) Generalized  $n$ -path parallel system:

We assume that each component  $X_i$  is on a separate parallel path and follows exponential distribution with installation time  $t_{0i}$ .

Since the probability of success of the parallel system should be greater than  $\rho$ , we have:

$$\rho \leq P(X_1)\psi_1 + P(X_2)\psi_2 + \dots + P(X_n)(1 - \sum_{i=1}^{n-1}(\psi_i)).$$

This in turn yields,

$$\rho \leq \sum_{i=1}^n \psi_i \exp \left( -\lambda_i(t - \hat{t}_{0i}) \right). \quad (11)$$

The inequation (11) can be solved for the following special case. (We are not aware of a closed-form general solution, but see below for an approximation.)

Assume that all components have identical failure rates and installation times:

1.  $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$ ; and
2.  $\hat{t}_{01} = \hat{t}_{02} = \dots = \hat{t}_{0n} = T_0$ .

This gives:

$$\rho \leq \exp \left( -\lambda(t - T_0) \right). \quad (12)$$

Taking the natural logarithm on both sides of inequation (12), we have:

$$t \leq T_0 - \frac{\ln \rho}{\lambda}. \quad (13)$$

Alternatively, we can attempt the general case by assuming a quadratic approximation for the exponential function, provided the values of  $\lambda_i$  are small enough that cubic and higher order terms can be disregarded. This is reasonable as each  $\lambda_i$  is supposed to be very small. Under this condition, for practically all values of time, the exponential function can be approximated by a quadratic expression.

Using (5) and (9), and since the probability of success of the system should be greater than  $\rho$ , we have:

$$\rho \leq \psi_1 \exp \left( -\lambda_1(t - \hat{t}_{01}) \right) + \psi_2 \exp \left( -\lambda_2(t - \hat{t}_{02}) \right) + \dots + \left( 1 - \sum_{i=1}^{n-1} \exp \left( -\lambda_i(t - \hat{t}_{0i}) \right) \right).$$

Therefore,

$$\begin{aligned} \rho \leq & \psi_1 \left( 1 - \lambda_1(t - \hat{t}_{01}) + \frac{\lambda_1^2(t - \hat{t}_{01})^2}{2!} + \dots \right) \\ & + \psi_2 \left( 1 - \lambda_2(t - \hat{t}_{02}) + \frac{\lambda_2^2(t - \hat{t}_{02})^2}{2!} + \dots \right) + \dots \\ & + \left( 1 - \sum_{i=1}^{n-1} \psi_i \right) \left( 1 - \lambda_n(t - \hat{t}_{0n}) + \frac{\lambda_n^2(t - \hat{t}_{0n})^2}{2!} + \dots \right), \end{aligned} \quad (14)$$

taking as before,  $\psi_n = (1 - \sum_{i=1}^{n-1} \psi_i)$ .

The inequality (14) can be evaluated as:

$$\begin{aligned} \rho \leq & \left( \frac{\sum_{i=1}^n (\psi_i (\lambda_i)^2)}{2} \right) t^2 - \left( \sum_{i=1}^n \psi_i \lambda_i^2 \hat{t}_{0i} + \sum_{i=1}^n \psi_i \lambda_i \right) t \\ & + \frac{\sum_{i=1}^n (\psi_i (\lambda_i)^2 (\hat{t}_{0i})^2)}{2} + \sum_{i=1}^n \psi_i \lambda_i \hat{t}_{0i} + 1 \end{aligned} \quad (15)$$

By solving the quadratic inequation (15) and finding roots, we can compute the region of time where the system is considered acceptable.

Let  $Q$  denote the determinant of the quadratic inequation.

$$\begin{aligned} Q &= \left( \sum_{i=1}^n \psi_i \lambda_i^2 \hat{t}_{0i} + \sum_{i=1}^n \psi_i \lambda_i \right)^2 \\ &- 2 \left( \sum_{i=1}^n (\lambda_i)^2 \psi_i \right) \left( 1 - \rho + \sum_{i=1}^n \psi_i \lambda_i \hat{t}_{0i} + \frac{\sum_{i=1}^n \psi_i (\lambda_i)^2 (\hat{t}_{0i})^2}{2} \right) \end{aligned}$$

It is evident that  $Q \geq 0$  for all real values of  $\psi_i, \lambda_i, \hat{t}_{0i} \forall i \in (1, 2, \dots, n)$ , and  $\rho$ .

The roots of the inequation are thus real and are given by:

$$\begin{aligned} t_1 &= \frac{\sum_{i=1}^n \psi_i \lambda_i^2 \hat{t}_{0i} + \sum_{i=1}^n \lambda_i \psi_i + \sqrt{Q}}{2 \sum_{i=1}^n \psi_i \lambda_i^2} \\ t_2 &= \frac{\sum_{i=1}^n \psi_i \lambda_i^2 \hat{t}_{0i} + \sum_{i=1}^n \lambda_i \psi_i - \sqrt{Q}}{2 \sum_{i=1}^n \psi_i \lambda_i^2} \end{aligned}$$

Using the obtained value of the roots, the temporal region where the system is acceptable can be obtained.

The Newton-Raphson method and other numerical analysis methods can also be used, and computer simulation can be used to calculate close results.

### 3.2.2 Mean Time To Failure (MTTF)

The MTTF of a machine is the average time in which the system may cease to work.

Given a system  $S$ , the MTTF is given by:

$$\text{MTTF} = \int_0^\infty P(S, t \geq T) dT \quad (16)$$

The formulae obtained are generalized and the one for parallel systems is different from that obtained using RBDs by Bazovsky [2] and Birolini [3], as our model is more general.

#### (i) Generalized $n$ -component series system

Without loss of generality, we assume that the installation time for each component is the same, given by  $T_0$ .

For a  $n$ -component series system, (2) holds. Using (9) and (16) and separating variables and constants, we get:

$$\text{MTTF} = \int_0^\infty \exp\left(-\left(\sum_{i=1}^n \lambda_i\right)T\right) \exp\left(\left(\sum_{i=1}^n \lambda_i\right)T_0\right) dT. \quad (17)$$

Since for all components the installation time is  $T_0$  and the function is valid for time  $T \geq T_0$ , here the second term  $\exp\left(\left(\sum_{i=1}^n \lambda_i\right)T_0\right)$  is a constant. Therefore, (17) can be simplified to:

$$\text{MTTF} = \frac{1}{\sum_{i=1}^n \lambda_i} \quad (18)$$

#### (ii) Generalized $n$ -path parallel system

We assume, as previously, that each parallel path has exactly one component  $X_i$ , and as before that each component follows exponential distribution and has installation time  $t_{0i}$ ,  $\forall i \in (1, 2, \dots, n)$ .

For a  $n$ -path parallel system, (5) holds.

Using (5), (9), and (16), we get:

$$\begin{aligned} \text{MTTF} = & \int_0^\infty \left( P(X_1, t \geq T) \psi_1 + P(X_2, t \geq T) \psi_2 + \dots \right. \\ & \left. + P(X_n, t \geq T) \left( 1 - \sum_{k=1}^{n-1} \psi_k \right) \right) dT. \end{aligned}$$

Since component  $X_i$  has installation time  $\hat{t}_{0i}$  and the function is valid for time  $t \geq \hat{t}_{0i}$ , we have:

$$\text{MTTF} = \sum_{i=1}^n \frac{\psi_i}{\lambda_i}. \quad (19)$$

### 3.2.3 Probability Density Function (pdf)

In probability theory, a pdf, or density of a continuous random variable, is a function that describes the relative likelihood for this random variable to occur at a given point. The probability for the random variable to fall within a particular region is given by the integral of this variable's density over the region.

Thus the pdf is given by differential of the cumulative distribution function.

For the exponential distribution,

$$f(T) = -\frac{d}{dT}P(S, t \geq T), \quad (20)$$

which means:

$$f(T) = \lambda \exp(-\lambda T). \quad (21)$$

The formulae obtained for this are general and different from those obtained using RBDs in Bazovsky [2] and Birolini [3].

#### (i) Generalized $n$ -component series system

For an  $n$ -component series system  $S$ , (2) holds.

With  $S$  given by  $S = X_1 \otimes X_2 \otimes X_3 \dots \otimes X_n$ , where  $X_1, X_2, \dots, X_n$  are components in series, using (20) and (2), we get:

$$f(T) = \left( \sum_{i=1}^n \lambda_i \right) \exp \left( - \sum_{i=1}^n \lambda_i (T - \hat{t}_{0i}) \right). \quad (22)$$

#### (ii) Generalized $n$ -path parallel system

We assume, as previously, that each parallel path has exactly one component  $X_i$ , which follows exponential distribution, and has installation time  $t_{0i}$ ,  $\forall i \in (1, 2, \dots, n)$ .

For an  $n$ -path parallel system, (5) holds.

With  $S$  given by  $X_1 \bar{\wedge}_{\psi_1} X_2 \bar{\wedge}_{\psi_2} X_3 \dots \bar{\wedge}_{\psi_{n-1}} X_n$ , using (20) and Corollary 2.2, we get:

$$f(T) = \sum_{i=1}^n \psi_i \lambda_i \exp \left( -\lambda_i (T - \hat{t}_{0i}) \right). \quad (23)$$

### 3.2.4 Mean Time Between Failures (MTBF)

Mean time between failures (MTBF) may be intuitively understood as the expected time between two successive failures of a system. It is given by the expectation of the pdf.

$$\text{MTBF} = \int_0^{\infty} t f(t) dt \quad (24)$$

#### (i) Generalized $n$ -component series system

For an  $n$ -component series system, (2) holds.

Using (24), assuming without loss of generality that  $T_0$  is the common installation time for all components  $X_i$  in the series system, we have:

$$\text{MTBF} = \int_{T_0}^{\infty} \left( \sum_{i=1}^n \lambda_i \right) \exp \left( -\sum_{i=1}^n \lambda_i (t - T_0) \right) t dt,$$

keeping in mind that the function is valid for time  $t \geq T_0$ . This reduces to:

$$\text{MTBF} = T_0 + \frac{1}{\sum_{i=1}^n \lambda_i}. \quad (25)$$

#### (ii) Generalized $n$ -path parallel system

We assume, as previously, that each parallel path has exactly one component  $X_i$ , and as before that each component follows exponential distribution and has installation time  $t_{0i}$ ,  $\forall i \in (1, 2, \dots, n)$ .

For an  $n$ -path parallel system, (5) holds. Using (16) and Corollary 2.2, we get:

$$\text{MTBF} = \int_0^{\infty} \sum_{i=1}^n \psi_i \lambda_i \exp(-\lambda_i (t - \hat{t}_{0i})) t dt.$$

Now each  $X_i$  has a pdf valid for  $t \geq \hat{t}_{0i}$ , so

$$\text{MTBF} = \sum_{i=1}^n \psi_i \left( \hat{t}_{0i} + \frac{1}{\lambda_i} \right) \quad (26)$$



### 3.2.5 Mean Time To Repair (MTTR)

The mean time to repair (MTTR) of a system refers to the average time required to repair a component that has failed or stopped working. As is common in prior literature, our analysis here excludes any consideration of extraneous delays (e.g., due to lack of availability of needed parts or equipment, or significant lead times to repair or install components).

The MTTR of a system is given by the difference of mean time between failures and mean time to failure:

$$\text{MTTR} = \text{MTBF} - \text{MTTF} \quad (27)$$

(i) Generalized  $n$ -component series system

For an  $n$ -component series system, using (18), (25), and (27), we get:

$$\text{MTTR} = T_0 + \frac{1}{\sum_{i=1}^n \lambda_i} - \frac{1}{\sum_{i=1}^n \lambda_i}$$

Thus the mean time to repair is given by:

$$\text{MTTR} = T_0.$$

(ii) Generalized  $n$ -path parallel system

For an  $n$ -path parallel system, using (19), (26), and (27), we get:

$$\text{MTTR} = \sum_{i=1}^n \psi_i \left( \hat{t}_{0i} + \frac{1}{\lambda_i} \right) - \sum_{i=1}^n \frac{\psi_i}{\lambda_i}$$

Thus, the MTTR is given by:

$$\text{MTTR} = \sum_{i=1}^n \psi_i \hat{t}_{0i}. \quad (28)$$

### 3.2.6 System Failure Rate (SFR) ( $\lambda_{eq}$ )

The failure rate for a single  $X_i$  is  $\lambda_i$  for the exponential distribution. We denote the system-wide equivalent failure rate as  $\lambda_{eq}$ , also with the exponential distribution. For  $\lambda_{eq} > 0$ , the probability of success of the system decays exponentially with time  $t \geq \hat{t}_{0i}$ . A larger value of  $\lambda_{eq}$  implies that the decay in success probability will be greater in a given amount of time, compared to a system with a smaller value.

The formulae obtained are more general and different from those obtained using RBDs in Bazovsky [2] and Birolini [3], again as a consequence of our more general model.

For an individual component  $X_i$ , the failure rate  $(\lambda_i)$  for the component can be obtained as:

$$\lambda_i = \frac{\frac{-d}{dT}P(X_i, t \geq T)}{P(X_i, t \geq T)}. \quad (29)$$

(i) Generalized  $n$ -component series system

For an  $n$ -component series system  $S$  of components  $X_i$ , (2) holds, which gives:

$$P(S, t \geq T) = \exp\left(-\sum_{i=1}^n (\lambda_i)T\right) \exp\left(\sum_{i=1}^n (\lambda_i \hat{t}_{0i})\right). \quad (30)$$

For the system  $S$ , evaluating  $\frac{-d}{dT}P(S, t \geq T)$ , we get:

$$\frac{-d}{dT}P(S, t \geq T) = \exp\left(\sum_{i=1}^n (\lambda_i \hat{t}_{0i})\right) - \left(\left(\sum_{i=1}^n \lambda_i\right)T\right)\left(\sum_{i=1}^n \lambda_i\right). \quad (31)$$

Using (29), we get:

$$\lambda_{eq} = \frac{\frac{-d}{dT}P(S, t \geq T)}{P(S, t \geq T)}.$$

This finally gives:

$$\lambda_{eq} = \sum_{i=1}^n \lambda_i. \quad (32)$$

(ii) Generalized  $n$ -path parallel system

We assume, as previously, that each parallel path has exactly one component  $X_i$  that follows the exponential distribution and has installation time  $t_{0i}$ .

For an  $n$ -path parallel system, (5) holds.

For a system  $S$  given by

$$S = X_1 \bar{\wedge}_{\psi_1} X_2 \bar{\wedge}_{\psi_2} X_3 \dots \bar{\wedge}_{\psi_{n-1}} X_n, \quad (33)$$

we get,

$$\frac{-d}{dT}P(S, t \geq T) = \sum_{i=1}^{n-1} \lambda_i \psi_i \exp\left(-\lambda_i(T - \hat{t}_{0i})\right). \quad (34)$$

We also have

$$\lambda_{eq} = \frac{\frac{-d}{dT}P(S, t \geq T)}{P(S, t \geq T)},$$

which gives:

$$\lambda_{eq} = \frac{\sum_{i=1}^n (m_i \lambda_i)}{\sum_{i=1}^n (m_i)}, \quad (35)$$

where  $m_i = \psi_i \exp(-\lambda_i(T - \hat{t}_{0i}))$ ,  $\forall i \in 1, 2, \dots, n$ .

The formula 35 indicates that the overall system failure rate is a weighted mean in a manner of speaking, where the weights of individual values  $\lambda_i$  are given by  $m_i$ .

### 3.3 Example

Many data center systems used for online hosting utilize power supply sources in parallel redundant design [7, 8]. The primary power supply comes from utility power. Sometimes, multiple power feeds to the facility are provided and are connected in parallel originating from independent power grids. A source of secondary power is a UPS battery system in the power room, and a third source of power is often a captive diesel generator, which may be capable of producing tens or hundreds of kilowatts of power for longer durations [4].

The three sources of power—utility power, UPS battery backup, and diesel generator set—form three dissimilar parallel paths in the data center power system. The standard definition of parallel systems fails in this model as the three paths are not equally likely and have quite different characteristics. While the utility power path is most likely to be chosen, the costliest path involving diesel generators is least likely to be taken, i.e., the path involving utility power is given highest priority of being chosen while the UPS battery path is given lower priority and diesel power path is given the least priority of being chosen [4].

Thus associating a non-probabilistic choice with such parallel paths would lead to erroneous results, and it is indeed known to be problematic to estimate MTBF and such parameters for data center power supplies [6]. Thus probabilistic parallelism comes into play.

For such a power supply system comprising utility power, UPS, and diesel generator, as described above, the probability of success can be given on the basis of Corollary 2.2 and (5) as:

$$P(X_1 \bar{\wedge}_{\psi_1} X_2 \bar{\wedge}_{\psi_2} X_3) = \psi_1 P(X_1) + \psi_2 P(X_2) + (1 - (\psi_1 + \psi_2))P(X_3),$$

where  $X_1$  represents the utility power component,  $X_2$  represents the UPS battery, and  $X_3$  represents the diesel generator path, such that  $\psi_1 > \psi_2 > \psi_3$ .

It is a simple matter, given data about such a system, to calculate the degradation parameters: the MTBF using (26), the MTTR using (28), the system failure rate using (35), etc.

## 4 Conclusion

In this paper, we have proposed an improved model to develop new formulae for degradation analysis parameters of repairable or maintainable probabilistic-choice systems. These parameters include MTTF, SFR, MTBF, MTTR, and pdf.

In the study of these parameters, each component in the system is assumed to follow an exponential distribution. The usage of probabilistic choice operators in the selection of parallel paths is motivated by the need to address more real world situations like sophisticated power supply systems for data centers, which cannot be modeled using RBDs and other classical approaches. The license for considering different installation times for components adds more flexibility to the model, which thus too better fits real-world systems where different components in a system may have different installation times.

Finally, we point out one important direction for future research. The model proposed can be extended in a straight-forward way to other distributions like Weibull, normal, etc., to develop formulae for parameters related to degradation analysis, along the lines we indicate in our approach, which assumed that components follow exponential distributions. This will help in analyzing systems which are composed of components which follow such different distributions.

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